B5440 – Exercise 1, Review and self-assessment

Exercises

1. Suppose $X, X_1, \ldots, X_n$ are independent and identically distributed with mean 0 and finite variance but *not* normal. The t-statistic to test the null hypothesis that $E(X) = 0$ is
$$\frac{\overline{X}_n}{S_n/\sqrt{n}},$$
where $\overline{X}_n$ is the sample mean and S_n is the sample standard deviation. Show that the t-statistic converges in distribution to a standard normal and note which named theorems you are using.

Solution:

First write
$$\frac{\overline{X}_n}{S_n/\sqrt{n}} = \sqrt{n}(\overline{X}_n/S_n - 0).$$
Then note that
$$\sqrt{n}(\overline{X}_n - 0) \to_d N(0, \sigma^2)$$
by the Central Limit Theorem, where $\sigma^2 = Var(X)$. Then
$$S_n^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}_n^2\right) \to_p 1(E(X^2) - (E(X))^2) = \sigma^2$$
by two applications of the weak law of large numbers and the continuous mapping theorem. By the continuous mapping theorem again, $S_n \to_p \sigma$. Finally by Slutsky's theorem
$$\frac{\overline{X}_n}{S_n/\sqrt{n}} \to_d N(0, \sigma^2)/\sigma =_d N(0, 1).$$

2. Suppose $X \sim$ exponential(θ) and $Y \sim$ exponential(η) with densities $f_\theta(x) = \theta e^{-\theta x}$, $f_\eta(y) = \eta e^{-\eta x}$. In the *uncensored* case, we observe both X and Y. In the *right censored* case we observe $(Z, \Delta) = (\min(X, Y), 1\{X \leq Y\})$.

(a) Densities.

i. Find the joint density of (X, Y).
ii. Find the joint density of (Z, Δ).

(b) Scores. Find the scores for θ and η:

i. in the uncensored case.
ii. in the right censored case.

(c) Information. Find the information for θ:

i. in the uncensored case.
ii. in the right censored case

Solution:

First in the uncensored case, the joint density is $f_\theta(x)f_\eta(y)$. To get the scores, we take logs and differentiate with respect to θ, η:

$$\log f_{x,y}(x, y; \theta, \eta) = \log f_\theta(x) + \log f_\eta(y) = \log\theta - \theta x + \log\eta - \eta y.$$

Then

$$\dot{l}_\theta(x, y) = \frac{\partial f_{x,y}}{\partial\theta} = \frac{1}{\theta} - x$$
$$\dot{l}_\eta(x, y) = \frac{\partial f_{x,y}}{\partial\eta} = \frac{1}{\eta} - y.$$

Since X and Y are independent, $E[(1/\theta - x)(1/\eta - y)] = 0$ and hence the information for θ is $E(\dot{l}_\theta(x, y)^2) =$

$$\frac{1}{\theta^2} - 2\frac{E(X)}{\theta} + E(X^2) = \frac{1}{\theta^2}.$$

Now in the censored case, the joint density of (Z, Δ) is

$$f_{Z,\Delta}(z, \delta; \theta, \eta) = \{(1 - F_\eta(z))f_\theta(z)\}^\delta\{(1 - F_\theta(z))f_\eta(z)\}^{1-\delta}$$

where F_η, F_θ are the cumulative distribution functions, e.g., $F_\theta(x) = 1 - e^{-\theta x}$. Then

$$l(z, \delta) = \log f_{Z,\Delta}(z, \delta; \theta, \eta) = \delta(-\eta z + \log\theta - \theta z) + (1 - \delta)(-\theta z + \log\eta - \eta z) =$$
$$\delta\log\theta - \theta z + \log\eta - \eta z - \delta\log\eta.$$

So the scores are

$$\dot{l}_\theta(z, \delta) = \frac{\delta}{\theta} - z$$
$$\dot{l}_\eta(z, \delta) = \frac{1 - \delta}{\eta} - z.$$

In this case the easier way to calculate the information for θ is using the second derivatives. It is easy to see that the off-diagonals of the Hessian matrix will be zero, since η does not appear in $\dot{l}_\theta$ and vice-versa. Hence the information for θ is

$$-E\left[\frac{\partial\dot{l}_\theta(Z, \Delta)}{\partial\theta}\right] = E[\Delta/\theta^2] =$$
$$P(\Delta = 1)/\theta^2.$$

First note that this will always be less than the information for θ in the uncensored case. We can then calculate how much by determining $P(\Delta = 1) =$

$$\begin{aligned} P(X \le Y) = E(P(X \le y \mid Y = y)) &= \\ \int_0^\infty \int_0^y \theta e^{-\theta x} \eta e^{-\eta y} \, dx dy &= \\ \int_0^\infty \eta e^{-\eta y} (1 - e^{-\theta y}) \, dy &= \\ 1 - \eta \int_0^\infty e^{-(\theta+\eta) y} \, dy &= \\ 1 - \frac{\eta}{\theta + \eta} &= \frac{\theta}{\theta + \eta}. \end{aligned}$$

3. If $X \ge 0$ and has distribution function F, show that

$$E(X) = \int_0^\infty (1 - F(x)) \, dx.$$

Solution:

$$\begin{aligned} \int_0^\infty (1 - F(x)) \, dx = \int_0^\infty \int_x^\infty dF(u) \, dx &= \\ \int_0^\infty \int_x^\infty dF(u) \, dx = \int_0^\infty \int_0^\infty 1[u > x] \, dF(u) \, dx &= \\ \int_0^\infty \int_0^\infty 1[u > x] \, dx \, dF(u) = \int_0^\infty u \, dF(u) &= E(X), \end{aligned}$$

using Fubini's theorem in the third line.