B5440 – Exercise 2, Processes and Martingales – Solutions

1. Answer and prove the following:

a. Let $X_1, X_2, \ldots$ be independent with mean 0. Is $M_n = X_1 + \cdots + X_n$ a martingale?

Solution: We check the definition:

$$E(M_n|\mathcal{F}_{n-1}) = E(X_1 + \ldots + X_n | X_1 + \ldots + X_{n-1}) = \\ E(X_n) + X_1 + \ldots + X_{n-1} = 0 + M_n.$$

Yes, it is a martingale.

b. Let $X_1, X_2, \ldots$ be independent with mean μ. Is $S_n = X_1 + \cdots + X_n$ a martingale? **Solution**: We check the definition:

$$E(M_n|\mathcal{F}_{n-1}) = E(X_1 + \ldots + X_n | X_1 + \ldots + X_{n-1}) = \\ E(X_n) + X_1 + \ldots + X_{n-1} = \mu + M_n.$$

So if $\mu \neq 0$, then it is **not** a martingale.

c. Let $X_1, X_2, \ldots$ be independent with mean 1. Is $M_n = X_1 \cdot X_2 \cdots X_n$ a martingale? **Solution**: We check the definition:

$$E(M_n|\mathcal{F}_{n-1}) = E(X_1 \cdot \ldots \cdot X_n | X_1 \cdot \ldots \cdot X_{n-1}) = \\ E(X_n) \cdot (X_1 \cdot \ldots \cdot X_{n-1}) = 1 \cdot M_n.$$

Yes, it is a martingale.

2. Show that if M is a martingale then so is the stopped process M^T.

Solution: First in the discrete time case, if M_n is a martingale then the stopped process $M_n^T = M_n$ if $n \leq T$ and $= M_T$ if $n > T$. We will use the fact that transformations of martingales by predictable processes are also martingales. So we need to find a transformation H_n that is predictable at time $n-1$ such that $M^T = H \bullet M$. It is $H_n = 1$ if $n \leq T$ and $H_n = 0$ if $n > T$. This is predictable at time $n-1$ because "today" we know whether we have reached T or not, thus "tomorrow's" value of H is known.

In continuous time we have

$$M^T(t) = \int_0^{t \wedge T} dM(u) = \int_0^t 1\{u \leq T\} \, dM(u)$$

which is a stochastic integral. The transformation $H(t) = 1\{t \leq T\}$ is predictable with respect to $\mathcal{F}_{t-}$ because it is adapted and it is left continuous. Therefore the stopped process is also a martingale.

3. Poisson process compensator:

Let $N(t)$ be the number of events in $[0, t]$ where $N(t) - N(s) \sim \text{Poisson}((t-s)\lambda)$ for $s < t$ and N has independent increments.

Does the Doob-Meyer decomposition apply to $N(t)$? If so, identify the compensator of $N(t)$. Also find the predictable variation process.

Solution

Problem statement: Find the unique predictable process $X(t)$ such that $N(t) - X(t)$ is a martingale.

Steps (WAG method):

1. I think $X(t) = \lambda t$.
2. Is λt predictable? Yes, it is left-continuous because

$$\lim_{s \uparrow t} \lambda s = \lambda t$$

   and all left-continuous processes are predictable.
3. Is $N(t) - \lambda t$ a martingale? Let's check the definition: for $s < t$, we have

$$\begin{aligned} E(N(t) - \lambda t | \mathcal{F}_s) = E(N(t) - N(s) + N(s) - \lambda t | \mathcal{F}_s) = \\ E(N(t) - N(s) | \mathcal{F}_s) + E(N(s) | \mathcal{F}_s) - \lambda t = \\ E(N(t) - N(s)) + N(s) - \lambda t = \\ (t-s)\lambda + N(s) - \lambda t = N(s) - \lambda s. \end{aligned}$$

Steps (start at the end method):

1. $N(t) - X(t)$ is a martingale iff $E(N(t) - X(t) | \mathcal{F}_s) = N(s) - X(s)$ for $s < t$ and a predictable process X. Hence,

$$\begin{aligned} E(N(t) - X(t) - N(s) + X(s) | \mathcal{F}_s) = 0 \Leftrightarrow \\ E((N(t) - N(s)) + (X(s) - X(t)) | \mathcal{F}_s) = 0 \Leftrightarrow \\ E(N(t) - N(s) | \mathcal{F}_s) = E(X(t) - X(s) | \mathcal{F}_s) \Leftrightarrow \\ (t-s)\lambda = E(X(t) - X(s) | \mathcal{F}_s). \end{aligned}$$

   Since $X(t)$ is predictable, we must have $E(X(t) - X(s) | \mathcal{F}_s) = X(t) - X(s)$. If not, then we could find a t^* such that $E(X(t^*) | \mathcal{F}_{t^*-}) \neq X(t^*)$, which would violate the definition of predictability. The result follows.

Predictable and optional variation processes

We just showed that $M(t) = N(t) - \lambda t$ is a martingale. Since the Poisson distribution has the same mean and variance, we can guess that $\langle M \rangle = \lambda t$.

By the Doob Meyer, and the fact that λt is predictable, we have to show that $M^2(t) - \lambda t$ is a martingale. We do this by checking the definition:

$$\begin{aligned} E((N(t) - \lambda t)^2 - \lambda t | \mathcal{F}_s) &= \\ E(N^2(t) - 2N(t)\lambda t + \lambda^2 t^2 - \lambda t | \mathcal{F}_s). \end{aligned}$$

Above we showed that $E(N(t)|\mathcal{F}_s) = N(s) + \lambda(t - s)$ by adding and subtracting $N(s)$, so we have now

$$\begin{aligned} E(N^2(t)|\mathcal{F}_s) - 2(N(s) + \lambda(t-s))\lambda t + \lambda^2 t^2 - \lambda t &= \\ E(N^2(t)|\mathcal{F}_s) - 2N(s)\lambda t - 2\lambda^2 t^2 + 2\lambda^2 st + \lambda^2 t^2 - \lambda t &= \\ E(N^2(t)|\mathcal{F}_s) - 2N(s)\lambda t - \lambda^2 t^2 + 2\lambda^2 st - \lambda t. \end{aligned}$$

Looking now at the first term, it equals

$$\begin{aligned} E((N(s) + N(t) - N(s))^2 | \mathcal{F}_s) &= \\ N(s)^2 + 2N(s)E(N(t) - N(s)) + E((N(t) - N(s))^2) &= \\ N(s)^2 + 2N(s)\lambda(t-s) + Var(N(t) - N(s)) + (E(N(t) - N(s)))^2 &= \\ N(s)^2 + 2N(s)\lambda(t-s) + \lambda(t-s) + \lambda^2(t-s)^2. \end{aligned}$$

Plugging back into the last line of the previous display:

$$\begin{aligned} N(s)^2 + 2N(s)\lambda(t-s) + \lambda(t-s) + \lambda^2(t-s)^2 - 2N(s)\lambda t - \lambda^2 t^2 + 2\lambda^2 st - \lambda t &= \\ N(s)^2 - 2N(s)\lambda s - \lambda s + \lambda^2(t-s)^2 - \lambda^2 t^2 + 2\lambda^2 st &= \\ N(s)^2 - 2N(s)\lambda s - \lambda s - \lambda^2 s^2 &= \\ (N(s) - \lambda s)^2 - \lambda s, \end{aligned}$$

which shows that the martingale property is satisfied.

What about $[M](t)$? For this we use the definition of the optional variation process, and observe that it equals

$$\begin{aligned} \sum_{s \le t} (M(s) - M(s-))^2 = \sum_{s \le t} (N(s) - N(s-) + \lambda s - \lambda s)^2 &= \\ \sum_{s \le t} (N(s) - N(s-))^2. \end{aligned}$$

Now if there is a jump at s the term in the sum equals 1, and if not, it equals 0. So this sum counts the number of events that occurred up to t, in other words it equals $N(t)$ itself. This is true for any counting process.